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A NOTE ON THE JOIN PROPERTY

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ABSTRACT. A Turing degree $\boldsymbol{a}$ satisfies the *join* property if, for every non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists $\boldsymbol{c} < \boldsymbol{a}$ with $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$. It was observed in [N4] that all degrees which are non-GL_2 satisfy the join property. This, however, leaves open many questions. Do all a.n.r. degrees satisfy the join property? How about the PA degrees or the Martin-Löf random degrees? A degree $\boldsymbol{b}$ satisfies the cupping property if, for every $\boldsymbol{a} > \boldsymbol{b}$, there exists $\boldsymbol{c} < \boldsymbol{a}$ with $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$. Is satisfying the cupping property equivalent to all degrees above satisfying join? We answer all of these questions by showing that above every low degree there is a low degree which does not satisfy join. We show, in fact, that all low fixed point free degrees $\boldsymbol{a}$ fail to satisfy join, and moreover, that the non-zero degree below $\boldsymbol{a}$ without any joining partner can be chosen to be a c.e. degree.

1. Introduction

The issue as to which Turing degrees satisfy the join property has long been of interest to researchers in computability theory. Robinson showed in his PhD thesis of 1972 [RR] that $\mathbf{0}'$ satisfies the join property, and this result was subsequently improved, by stages, in a number of directions. Greenberg, Montalbán and Shore [GMS] have shown that actually all generalized high degrees satisfy the *complementation property*, where a degree $\boldsymbol{a}$ is generalized high if $\boldsymbol{a}' = (\boldsymbol{a} \vee \mathbf{0}')'$, and where $\boldsymbol{a}$ satisfies the complementation property if, for every non-zero $\boldsymbol{b} < \boldsymbol{a}$, there exists $\boldsymbol{c}$ with $\boldsymbol{b} \vee \boldsymbol{c} = \boldsymbol{a}$ and $\boldsymbol{b} \wedge \boldsymbol{c} = \mathbf{0}$. It was shown in [N4] that all degrees which are non-GL_2 satisfy the join property, where a degree $\boldsymbol{a}$ is GL_n if $\boldsymbol{a}^{(n)} = (\boldsymbol{a} \vee \mathbf{0}')^{(n-1)}$ and is L_n if $\boldsymbol{a} < \mathbf{0}'$ and $\boldsymbol{a}^{(n)} = \mathbf{0}^{(n)}$. Since initial segment results, see for example [ML], suffice to show that there are degrees which are L_1 which do not satisfy the join property, and also degrees which are L_2 but not L_1 of this kind, the latter result is sharp in terms of the jump hierarchy.

Every degree which is not above $\mathbf{0}'$, however, is bounded by a degree which is GL_1, and so the degrees which are non-GL_2 are not upward closed. In looking to understand for which degrees it is the case that all degrees in the upper cone satisfy join, it becomes natural to ask whether all PA degrees or all a.n.r. degrees satisfy join, because these are upward closed classes which have played significant roles in degree theory. Recall that degree is PA if it contains a set which effectively codes a complete and consistent extension of Peano Arithmetic. The a.n.r. (array non-recursive) degrees were defined by Downey, Jockusch and Stob [DJS]; a degree

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is a.n.r. if there is no function which is wtt-reducible to $\emptyset'$ and dominates all $\boldsymbol{a}$-computable functions. If we wish to avoid explicit mention of wtt reductions in this definition, then we may observe that this is equivalent to requiring that $\boldsymbol{a}$ contains a function which is not dominated by the modulus of convergence function for some computable enumeration of the halting problem. Both of these classes have been shown to satisfy the cupping property [DJS], [AK]. Another natural question is as to whether satisfaction of the cupping property is equivalent to all degrees above satisfying join. The result of this paper answers all of these questions.

We let $\{\Psi_n\}_{n\in\omega}$ be an effective listing of all Turing functionals. A degree is *low* if it is L_1. A degree is fixed point free if it contains a function f such that, for all n, $\Psi_n \neq \Psi_{f(n)}$. A function f is diagonally non-computable (d.n.c.) if, for all n, $f(n) \neq \Psi_n(\emptyset; n)$. Jockusch has shown that a degree is fixed point free iff it contains a d.n.c. function [JS].

Theorem 1.1. *Every low fixed point free degree fails to satisfy the join property. More specifically, if* $\mathbf{a}$ *is low and fixed point free, then there exists a promptly simple degree* $\mathbf{b} < \mathbf{a}$ *such that no degree* $\mathbf{d} < \mathbf{a}$ *joins* $\mathbf{b}$ *up to* $\mathbf{a}$*.*

Corollary 1.2. Above every low degree there is a low degree which does not satisfy the join property.

Proof. Given any low degree $\mathbf{c}$ consider $\mathbf{a} > \mathbf{c}$ which is both low over $\mathbf{c}$ and PA over it. The existence of such a degree $\mathbf{a}$ follows from the low basis theorem [JS] relativized to $\mathbf{c}$. Any degree which is PA over $\mathbf{c}$ is PA, and therefore fixed point free. □

Corollary 1.3. There exist a.n.r. degrees which do not satisfy the join property.

Proof. There are a.n.r. degrees which are low [DJS], and it follows immediately from the definition that the a.n.r. degrees form an upward closed class. □

Corollary 1.4. There exist PA degrees which do not satisfy the join property.

Proof. It follows from the low basis theorem [JS] that there exist PA degrees which are low. All PA degrees are fixed point free. □

Corollary 1.5. There exist Martin-Löf random degrees which do not satisfy the join property.

Proof. As for Corollary 1.4. □

Corollary 1.6. Satisfaction of the cupping property is not equivalent to all degrees above satisfying join.

Proof. Take a low PA degree $\boldsymbol{a}$ and then a degree $\boldsymbol{b}$ which is low over $\boldsymbol{a}$ and PA. By Theorem 1.1 this degree $\boldsymbol{b}$ fails to satisfy the join property, but $\boldsymbol{a}$ satisfies the cupping property, since all PA degrees do so. □

This leaves open two significant questions:

Question 1.7. Can $\mathbf{0}'$ be defined as the least degree such that all degrees above it satisfy join?

Question 1.8. In $\mathcal{D}[\leq \mathbf{0}']$, can L_2 be defined as those degrees for which it is not the case that all degrees above satisfy join?

Our notation and terminology will be standard, and will follow [RS] unless explicitly stated otherwise. We shall use the variables i, j, k, n, m, s, t for elements of ω; β, δ for elements of $2^{<\omega}$; α for elements of $\omega^{<\omega}$; f, g, h for total functions $\omega^n \mapsto \omega$; υ for partial functions $\omega \mapsto \omega$; A, B, C, D for subsets of ω. We let W_i denote the ith c.e. set. $\Psi(A; n)$ denotes the output of the Turing functional Ψ on argument n, given oracle input A. We shall assume that, for any β, i, n, m, $\Psi_i(\beta; n) \downarrow = m$ only if this computation converges in less than $|\beta|/2$ steps and $\Psi_i(\beta; n') \downarrow$ for all $n' < n$ (we consider $|\beta|/2$ here because often we shall consider oracles of the form $A \oplus B$, where the $\oplus$ function is defined subsequently). A c.e. set B is *promptly simple* if it has infinite complement and there exists a computable function g and a computable enumeration $\{B_s\}_{s\in\omega}$ such that whenever W_i is infinite, $\exists s \exists n[n \in W_{i,s} - W_{i,s-1} \ \& \ n \in B_{g(s)}]$. We let $A \restriction n$ denote the initial segment of A of length n. For (possibly) partial functions υ_0, υ_1, we let $\upsilon_0 \oplus \upsilon_1$ be the (possibly) partial function υ such that $\upsilon(2n) = \upsilon_0(n)$ if this value is defined and is undefined otherwise, $\upsilon(2n+1) = \upsilon_1(n)$ if this value is defined and is undefined otherwise. At any point during a construction, by a *large* number, we mean a number larger than any previously mentioned during the construction.

2. The proof of Theorem 1.1

We consider first a proof of the result due to Slaman and Steel [SS], and independently due to Cooper [BC], that there exist non-zero c.e. degrees $\mathbf{b} < \mathbf{a}$ such that no Δ^0_2 degree $\mathbf{d} < \mathbf{a}$ joins $\mathbf{b}$ up to $\mathbf{a}$. Actually we shall build $\mathbf{b}$ which is promptly simple, since there is really no extra complication involved in doing so. We construct c.e. sets B and C such that the following requirements are satisfied:

$\mathcal{P}_i$: If W_i is infinite then $\exists s \exists n[n \in W_{i,s} - W_{i,s-1} \ \& \ n \in B_s]$;
$\mathcal{Q}_i$: If $\Phi_i(B \oplus C)$ ($= D_i$ say) is total and $\Psi_i(B \oplus D_i) = C$ then $\Gamma_i(D_i) = B$;

where $\{(\Phi_i, \Psi_i)\}_{i\in\omega}$ is an effective listing of all pairs of Turing functionals, and each Γ_i is a functional that we build during the course of the construction. B will have infinite complement by construction. In what follows we will often abuse notation by writing D_i in order to denote $\Phi_i(B \oplus C)$, even though this may actually be a partial function.

We shall have one strategy for each requirement, and the requirements and their corresponding strategies are prioritized $\mathcal{P}_0, \mathcal{Q}_0, \mathcal{P}_1, \mathcal{Q}_1 \cdots$. At each stage s of the construction we perform another step of the instructions for each of the first s strategies in turn. The construction will have finite injury. Since the strategies are very simple and the interactions between them are not complicated, it is probably easiest just to describe the strategies directly and then verify that they work.
In what follows we adopt the standard convention that when discussing any point in the construction, we may write B, C etc in order to denote their present values. At any point during the construction we let δ_i denote the finite string $\Phi_i(B \oplus C)$ (which we can assume to be binary).

The $\mathcal{Q}_i$ strategy. The basic idea behind this strategy is extremely simple – at each stage at which it is run the strategy considers the least n such that $\Gamma_i(\delta_i; n) \uparrow$ and it looks to define this value. There are a few small considerations that have to be made, however.

(a) Whenever we enumerate an axiom defining a value $\Gamma_i(\delta; n)$, we wish to ensure that it is already the case $\Gamma_i(\delta; n') \downarrow$ for all $n' < n$.
(b) When we enumerate such an axiom, we wish to ensure that δ and the present value B already map via Ψ_i to an initial segment of C which is longer than n. The motivation for this is in order to co-ordinate with the $\mathcal{P}$ strategies, which will be using agitators in C.
(c) We also have to make sure that the axioms enumerated are consistent. We do not wish to enumerate an axiom $\Gamma_i(\delta; n) = 1$, for example, when there is already some $\delta' \supseteq \delta$ for which we have enumerated the axiom $\Gamma_i(\delta'; n) = 0$. The precise instructions are therefore as below.

At any point in the construction the length of agreement for $\mathcal{Q}_i$ is the greatest m such that $C \upharpoonright m \subseteq \Psi_i(B \oplus \delta_i)$. At each stage at which it is run the strategy considers the least n such that $\Gamma_i(\delta_i; n) \uparrow$ and if n is less than the length of agreement then it proceeds as follows. Let β and δ be, respectively, the initial segments of B and δ_i used in the computation $\Psi_i(B \oplus \delta_i; n) = C(n)$. If there exists a shortest δ' with $\delta \subseteq \delta' \subseteq \delta_i$ such that $\Gamma_i(\delta'; n') \downarrow$ for all $n' < n$ and for which it is consistent with all axioms previously enumerated to enumerate the axiom $\Gamma_i(\delta'; n) = \beta(n)$, then enumerate this axiom.

If $\Phi_i(B \oplus C) = D_i$ is total and $\Psi_i(B \oplus D_i) = C$ then it is not hard to see that these instructions ensure that Γ_i is total. In order to ensure that $\Gamma_i(D_i) = B$ we just have to make sure that:

($\star_i$) when $n \in B$, D_i does not extend any string δ for which we enumerate an axiom $\Gamma_i(\delta; n) = 0$.

The $\mathcal{P}_i$ strategy. When this strategy enumerates some n into B it must be able to ensure for each $\mathcal{Q}_{i'}$ of higher priority, that if $D_{i'}$ is total and $\Psi_{i'}(B \oplus D_{i'}) = C$, then condition ($\star_{i'}$) is not violated, i.e. $D_{i'}$ cannot extend any string δ for which we have enumerated an axiom $\Gamma_{i'}(\delta; n) = 0$. In order to achieve this the strategy uses a standard agitator technique: it will not enumerate n into B, unless there exists $\beta = B \upharpoonright n'$ for some $n' < n$ and there exists m such that, for all δ and $i' < i$ for which we have enumerated an axiom $\Gamma_{i'}(\delta; n) = 0$, it is the case $\Psi_{i'}(\beta \oplus \delta; m) = 0$. Then, upon enumerating n into B, the strategy enumerates m into C and looks to preserve β as an initial segment of B. This latter task it attempts to achieve simply by initializing all lower priority strategies – these strategies will only be allowed to enumerate numbers into B or C which are greater than the last stage at which they were initialized. Any higher priority strategy $\mathcal{P}_{i''}$ may subsequently enumerate a number into B which is less than $|\beta|$, but then *this* strategy will similarly ensure that no problematic δ can be an initial segment of $D_{i'}$ if it is to be the case that $D_{i'}$ is total and $\Psi_{i'}(B \oplus D_{i'}) = C$ (and if $i'' > i'$), since it will also enumerate some m' into C and preserve a (shorter) initial segment of B. The instructions for the strategy are as follows.

The instructions for $\mathcal{P}_i$. When the strategy is first run (subsequent to any initialization) it chooses a large agitator m and then a large marker $n > m$. At every subsequent stage s at which it has not yet been declared satisfied it performs the following steps:

(1) Check to see whether there exists any $i' < i$ for which the strategy has not previously *seen convergence* (this will initially be all $i' < i$) and for which $\Gamma_{i'}(\delta_{i'}; m) \downarrow$. If so then initialize all lower priority strategies, redefine n to be large, declare that the strategy has seen convergence for each $i' < i$ for which $\Gamma_{i'}(\delta_{i'}; m) \downarrow$, and perform no further action at this stage. If not then proceed to (2).
(2) If there exists n' such that $n < n'$ and $n' \in W_{i,s} - W_{i,s-1}$, then enumerate n' into B, enumerate m into C, declare the strategy to be satisfied and initialize all lower priority strategies.

The verification. It is clear that each strategy is initialized a finite number of times, and since each marker for a strategy $\mathcal{P}_i$ is only redefined a finite number of times it follows that each $\mathcal{P}_i$ requirement is satisfied. That the complement of B is infinite follows since markers are chosen to be large. In order to show that the $\mathcal{Q}_i$ requirement is satisfied, let s_0 be such that the $\mathcal{Q}_i$ strategy is never initialized after stage s_0. It suffices to show that if D_i is total and the length of agreement is unbounded, then D_i is not compatible with any δ for which we enumerate an axiom subsequent to stage s_0, of the form $\Gamma_i(\delta; n) = 0$ for some n which is enumerated into B. This suffices, because then the instructions for $\mathcal{Q}_i$ clearly ensure that $\Gamma_i(D_i)$ is total.

So suppose that n is enumerated into B by a strategy $\mathcal{P}_{i'}$ for $i' > i$ at a stage $s_1 > s_0$ and that, prior to this enumeration, we have enumerated an axiom $\Gamma_i(\delta; n) = 0$. Let i'' be the least such that $\mathcal{P}_{i''}$ enumerates a number into B at a stage $s_2 \geq s_1$, and such that $\mathcal{P}_{i''}$ is not initialized at any stage in the interval $[s_1, s_2]$ (so that $i'' \leq i'$). Then $\mathcal{P}_{i''}$ has already seen convergence for i when the axiom $\Gamma_i(\delta; n) = 0$ is enumerated, since its agitator m is less than or equal to that of $\mathcal{P}_{i'}$. Let β and δ' be, respectively, the initial segments of B and δ_i used in the computation $\Psi_i(B \oplus \delta_i; m)$ when $\mathcal{P}_{i''}$ saw convergence for i. Since $\mathcal{P}_{i''}$ initialized all lower priority strategies when it saw convergence for i, $\delta' \subseteq \delta$, and β is an initial segment of the final value B – in order to see this note that δ_i cannot change below any given length unless B or C change below the relevant use and that once lower priority strategies are initialized, they will subsequently choose markers and agitators larger than all previously observed uses. It follows that if the length of agreement is unbounded then δ cannot be an initial segment of D_i, because $\Psi_i(\beta \oplus \delta'; m) = 0$ and $\mathcal{P}_{i''}$ enumerates m into C.

Proving the full theorem. In order to prove the full theorem we consider a version of the construction just described. Now, however, we suppose that we are given an approximation $\{f_s\}_{s\in\omega}$ to f which is of low degree and is a d.n.c. function. We consider $\Phi_i(f)$ rather than $\Phi_i(B \oplus C)$, and we must ensure that $B \oplus C$ is computable in f. The requirements $\mathcal{P}_i$ are just as before. The new $\mathcal{Q}_i$ requirements are:

$\mathcal{Q}_i$: If $\Phi_i(f)$ ($= D_i$ say) is total and $\Psi_i(B \oplus D_i) = C$ then $\Gamma_i(D_i) = B$;

So we still construct B and C and these sets will play the same roles as before – B will be the set whose degree doesn't have a joining partner, and C provides the agitators which are used in order to satisfy the requirements $\mathcal{P}_i$. Once again we adopt the standard convention that when discussing any point in the construction, we may write B, C, f etc., in order to denote their present values. At any point during the construction we now let δ_i denote the finite binary string $\Phi_i(f)$. We use Kučera's technique of fixed point free permitting to ensure that B and C are computable in f.

The potential problem now, is that when $\mathcal{P}_i$ sees convergence for $i' < i$ because $\Gamma_{i'}(\delta; m)\downarrow$ for some $\delta \subseteq \delta_{i'}$, it will initialize all lower priority strategies as before, but now $\delta_{i'}$ may become incompatible with δ at subsequent stages. If at such a subsequent stage it sees convergence for i' again, then it must once again initialize all lower priority strategies, because the initial segment of B that it needs to preserve may now be longer. This might happen an infinite number of times, causing $\mathcal{P}_i$ to initialize lower priority strategies an infinite number of times. In order to avoid this problem, we use the fact that f is of low degree. The construction specifies Turing functionals which compute B and C from f. By the recursion theorem we may assume we are given indices for these reductions in advance, and thus, for any m and i', we can approximate during the construction whether or not there exist α, β which are initial segments of (the final values) f and B respectively, such that if $\delta = \Phi_{i'}(\alpha)$ then $C \upharpoonright (m+1) \subseteq \Psi_{i'}(\beta \oplus \delta)$. We only define $\Gamma_{i'}$ on argument m when our approximation tells us that such initial segments exist. This, then, ensures that $\mathcal{P}_i$ only initializes strategies of lower priority finitely many times.

The terminology we use is the same as that used in the section on d.n.c. functions in Nies' book [AN], which gives a very clear introduction to fixed point free permitting.

The details are as follows. During the construction we build a partial computable function υ. By the recursion theorem we may assume that we are given an index for υ and so a computable function g such that for all i, $\upsilon(i) \simeq \Psi_{g(i)}(\emptyset; g(i))$. By the recursion theorem we may also assume that we are given a computable function h such that, for all m and i, $\lim_s h(i,m,s)$ is equal to 1 if there exist α, β which are initial segments of f and B respectively, such that if $\delta = \Phi_i(\alpha)$ then $C \upharpoonright (m+1) \subseteq \Psi_i(\beta \oplus \delta)$, and is equal to 0 otherwise. As before, the length of agreement for $\mathcal{Q}_i$ at any point during the construction is the greatest m such that $C \upharpoonright m \subseteq \Psi_i(B \oplus \delta_i)$. If $\Gamma_i(\delta_i; m)\downarrow$ at any stage, then the *initial segment of* δ_i *used on argument* m is the shortest $\delta \subseteq \delta_i$ such that $\Gamma_i(\delta; m)\downarrow$. The modified strategies work as follows.

The $\mathcal{Q}_i$ strategy. At each stage at which it is run the strategy considers the least n such that $\Gamma_i(\delta_i; n)\uparrow$ and if n is less than the length of agreement and $h(i,n,s) = 1$ then it proceeds as follows. Let β and δ be, respectively, the initial segments of B and δ_i used in the computation $\Psi_i(B \oplus \delta_i; n) = C(n)$. If there exists a shortest δ' with $\delta \subseteq \delta' \subseteq \delta_i$ such that $\Gamma_i(\delta'; n')\downarrow$ for all $n' < n$ and for which it is consistent with all axioms previously enumerated to enumerate the axiom $\Gamma_i(\delta'; n) = \beta(n)$, then enumerate this axiom.

The $\mathcal{P}_i$ strategy. Once it has been declared satisfied, the strategy performs no further actions even if subsequently initialized, since then the requirement has been permanently satisfied. When the strategy is first run (subsequent to any initialization) it chooses a large agitator m and then a large marker $n > m$. At every subsequent stage s at which it has not yet been declared satisfied it performs the following steps:

(1) Check to see whether, for all t with $m \leq t \leq s$, $f_t(g(i)) = f_s(g(i))$. If not then initialize this strategy together with all strategies of lower priority, performing no more actions at this stage. Otherwise proceed to (2).
(2) Check to see whether there exists any $i' < i$ for which the strategy has not previously seen convergence for any (i', δ) such that δ is an initial segment of the present value $\delta_{i'}$, and such that $\Gamma_{i'}(\delta_{i'}; m) \downarrow$. If so then perform steps (a) to (c) below, performing no further action at this stage. If not then proceed to (3).
   (a) Initialize all lower priority strategies;
   (b) Redefine n to be large;
   (c) For each $i' < i$ such that $\Gamma_{i'}(\delta_{i'}; m) \downarrow$, let δ be the initial segment of $\delta_{i'}$ used on argument m, and declare that the strategy has seen convergence for (i', δ).
(3) If there exists n' such that $n < n'$ and $n' \in W_{i,s} - W_{i,s-1}$, then define $\upsilon(i) = f_s(g(i))$, enumerate n' into B, enumerate m into C, declare the strategy to be satisfied and initialize all lower priority strategies.

The verification. First we show that $B \oplus C$ is computable in f. Since C computes B by simple permitting, it suffices to show that C is computable in f. Given input m, use f to compute $t > m$ such that for all $i \leq m$, $f_t(g(i)) = f(g(i))$. Then m is in C only if it has been enumerated in by stage t, since if we enumerate m into C at a stage $s > t$ then this is done by a strategy $\mathcal{P}_i$ for $i < m$ and this strategy would have been initialized (and so be forced to choose an agitator greater than m) unless for all t' with $m \leq t' \leq s$, $f_{t'}(g(i)) = f_s(g(i))$. Thus the value $\upsilon(i) = \Psi_{g(i)}(\emptyset; g(i)) = f_s(g(i))$ we define at stage s equals $f(g(i))$ and therefore f is not a d.n.c. function, which gives a contradiction.

Next we must show that, for any n, i, there are only a finite number of δ for which we enumerate an axiom $\Gamma_i(\delta; n) = j$ (for any $j \in \{0, 1\}$). If $\lim_s h(i, n, s) = 0$ then this is clear, so suppose $\lim_s h(i, n, s) = 1$. Let δ and β be the shortest initial segments of the final values of D_i and B such that $\Psi_i(\beta \oplus \delta) \supseteq \tau$, where τ is $C \upharpoonright (n + 1)$. If $C(n) = 0$ then the result is clear because then it will always be consistent to enumerate the axiom $\Gamma_i(\delta; n) = 0$, so suppose otherwise. Let m be the length of the longest string δ_0 for which we enumerate an axiom $\Gamma_i(\delta_0; n) = 0$, and let $m' = \max\{|\delta|, |\delta_0|\}$. Then at stages s such that $\beta \subset B_s$ we do not enumerate any axiom $\Gamma_i(\delta_1; n) = 1$ for $\delta_1 \supseteq \delta$ of length greater than m'.

It follows that each strategy is initialized only finitely often and that the marker for each $\mathcal{P}_i$ requirement is redefined only a finite number of times. Therefore each $\mathcal{P}_i$ requirement is satisfied. That the complement of B is infinite follows since markers are chosen to be large. That the requirement $\mathcal{Q}_i$ is satisfied follows from almost exactly the same argument as before. Let s_0 be such that the $\mathcal{Q}_i$ strategy is never initialized after stage s_0. Suppose that n is enumerated into B by a strategy $\mathcal{P}_{i'}$ for $i' > i$ at a stage $s_1 > s_0$ and that, prior to this enumeration, we have enumerated an axiom $\Gamma_i(\delta; n) = 0$. When this axiom is enumerated, $\mathcal{P}_{i'}$ has seen convergence for some (i, δ_0) such that $\delta_0 \subseteq \delta$. Let i'' be the least such that $\mathcal{P}_{i''}$ enumerates a number into B at a stage $s_2 \geq s_1$, and such that $\mathcal{P}_{i''}$ is not initialized at any stage in the interval $[s_1, s_2]$ (so that $i'' \leq i'$). Then $\mathcal{P}_{i''}$ has already seen convergence for some (i, δ_1) such that $\delta_1 \subseteq \delta_0$ when the axiom $\Gamma_i(\delta; n) = 0$ is enumerated, since its agitator m is less than or equal to that of $\mathcal{P}_{i'}$. Let β be the initial segment of B used in the computation $\Psi_i(B \oplus \delta_i; m)$ when $\mathcal{P}_{i''}$ saw convergence for (i, δ_1). Since

$\mathcal{P}_{i''}$ initialized all lower priority strategies when it saw convergence for (i, δ_1), β is an initial segment of the final value B. It follows that if the length of agreement is unbounded then δ cannot be an initial segment of D_i, because $\Psi_i(\beta \oplus \delta_1; m) = 0$ and $\mathcal{P}_{i''}$ enumerates m into C.

Final remarks. The proof given here actually suffices to show that every low c.e. degree is bounded by a low c.e. degree which does not satisfy join. Given W of low c.e. degree, let f be d.n.c. of low degree above that of W. The construction ensures that $W \oplus B \oplus C$ does not satisfy the join property.

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